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DETERMINING THE ELASTIC CHARACTERISTICS OF HOMOGENEOUS ANISOTROPIC BODIES

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As is generally known [1], in an arbitrary system of coordinates $x_i(i = 1, 2, 3)$, Hooke's Law for an anisotropic body has the form

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn} \quad (k, l = 1, 2, 3) \tag{1}$$

or

$$\sigma_{kl} = b_{klmn}\varepsilon_{mn} \quad (k, l = 1, 2, 3), \tag{2}$$

where ε_{kl}, σ_{kl}, a_{klmn}, b_{klmn} are the components of the deformation, tension, elastic-pliability, and elastic-modulus tensors, respectively. In (1) and (2) the usual rule of summation over repeating indices is assumed.

In the general case (of a body of a triclinic system), as a result of the known conditions of symmetry, the tensors a_{klmn} and b_{klmn} have 21 different components. In the coordinate system associated with the major axes of anisotropy, this number is equal to 18, and in the presence of a single plane of symmetry (the body of a monoclinic system) it is as high as 9, etc. However, if the orientation of the major axes of anisotropy and the degree of symmetry of the body in question are not known beforehand, the determination of all of its elastic characteristics on the basis of the standard uniaxial-tension (compression) experiments and torsion in the directions of the coordinate axes encounters significant difficulties [2]. This is caused by the fact that in each such experiment it is necessary to measure all of the deformation components. The considerations adduced in [1, pp. 150-151] on finding this tension tensor, which corresponds to a spherical deformation tensor and determines the major axes of anisotropy, can scarcely be realized in practice.

In experimental mechanics the methods of holographic and laser speckle interferometry [3] are currently being applied more and more widely, permitting accurate determination of the components of the vector of the shifts of the surface of a deformed body. Moreover, if the external loads are known, then in order to find a_{klmn} (or b_{klmn}) it is possible to obtain a closed system of equations. The present article is devoted to construction of the latter.

Let us consider an elastic homogeneous body of volume v with a surface S, to which external loads p_k are applied, causing shifts u_k (k = 1, 2, 3) on S, which are assumed to be known as a result of the appropriate measurements. From this information it is necessary to determine all of the components of the a_{klmn} (or b_{klmn}) tensor.

Let us examine the moments of the zeroth and first orders for tension and deformation:

$$\begin{gathered} M^0_{kl} = \int_v \sigma_{kl} dv, \quad N^0_{kl} = \int_v \varepsilon_{kl} dv, \\ M^i_{kl} = \int_v \sigma_{kl} x_i dv, \quad N^i_{kl} = \int_v \varepsilon_{kl} x_i dv \quad (k, l, i = 1, 2, 3). \end{gathered} \tag{3}$$

From (1) and (3) it follows that

$$N^i_{kl} = a_{klmn} M^i_{mn} \quad (k, l = 1, 2, 3, \ i = 0, 1, 2, 3). \tag{4}$$

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Let us designate

$$Q_k^{s_1 s_2 s_3} = \int_S x_1^{s_1} x_2^{s_2} x_3^{s_3} p_k dS, \quad U_{kl}^0 = \int_S u_k n_l dS,$$

$$U_{kl}^i = \int_S u_k n_l x_i dS \quad (k, l, i = 1, 2, 3), \tag{5}$$

where s_1, s_2, s_3 are whole nonnegative numbers; and n_k are the components of the unit vector of the normal external to S.

From the equilibrium equations (there are no mass forces) and the Ostrogradskii–Gauss formula, we obtain the equalities [4]

$$\int_v x_1^{s_1} x_2^{s_2} x_3^{s_3} \frac{s_l}{x_l} \sigma_{kl} dv = Q_k^{s_1 s_2 s_3} \quad (k = 1, 2, 3). \tag{6}$$

In particular, for $n \equiv s_1 + s_2 + s_3 = 0$ (i.e. $s_1 = s_2 = s_3 = 0$), from (6) we have $Q_k^{000} = 0$ (k = 1, 2, 3), which corresponds to the main vector of the external forces being equal to zero. For n = 1, expressions for the moments of the tensions of the zeroth order through external loads follow from (6):

$$M_{k1}^0 = Q_k^{100}, \; M_{k2}^0 = Q_k^{010}, \; M_{k3}^0 = Q_k^{001} \quad (k = 1, 2, 3). \tag{7}$$

As a result of the symmetry of the components of the tension tensor, the conditions for the main moment of the external forces being equal to zero derive from (7): $Q_3^{010} - Q_2^{001} = 0$, $Q_1^{001} - Q_3^{100} = 0$, $Q_2^{100} - Q_1^{010} = 0$.

For n = 2, from (6) we obtain a system of 18 equations for finding all 18 tension moments of the first order [4], from which we have

$$\begin{aligned}
&M_{11}^1 = \frac{1}{2} Q_1^{200}, \; M_{11}^2 = Q_1^{110} - \frac{1}{2} Q_2^{200}, \; M_{11}^3 = Q_1^{101} - \frac{1}{2} Q_3^{200}, \\
&M_{22}^1 = Q_2^{110} - \frac{1}{2} Q_1^{020}, \; M_{22}^2 = \frac{1}{2} Q_2^{020}, \; M_{22}^3 = Q_2^{011} - \frac{1}{2} Q_3^{020}, \\
&M_{33}^1 = Q_3^{101} - \frac{1}{2} Q_1^{002}, \; M_{33}^2 = Q_3^{011} - \frac{1}{2} Q_2^{002}, \; M_{33}^3 = \frac{1}{2} Q_3^{002}, \\
&M_{12}^1 = \frac{1}{2} Q_2^{200}, \; M_{12}^2 = \frac{1}{2} Q_1^{020}, \; M_{12}^3 = \frac{1}{2} (Q_1^{011} + Q_2^{101} - Q_3^{110}), \\
&M_{13}^1 = \frac{1}{2} Q_3^{200}, \; M_{13}^2 = \frac{1}{2} (Q_1^{011} + Q_3^{110} - Q_2^{101}), \; M_{13}^3 = \frac{1}{2} Q_1^{002}, \\
&M_{23}^1 = \frac{1}{2}(Q_2^{101} + Q_3^{110} - Q_1^{011}), \; M_{23}^2 = \frac{1}{2} Q_3^{020}, \; M_{23}^3 = \frac{1}{2} Q_2^{002}.
\end{aligned} \tag{8}$$

It is necessary to note that for $n \geq 3$ the number of equations (6) is less than the number of unknowns, and so it is impossible to obtain expressions for all of the tension moments of the second and higher orders due to external loads [4].

Let us now examine the deformation moments. From the Cauchy relations connecting ε_{kl} and u_k, and the Ostrogradskii–Gauss formula, we have

$$\int_v \varepsilon_{kl} dv = \frac{1}{2} \int_v (u_{k,l} + u_{l,k}) dv = \frac{1}{2} \int_S (u_k n_l + u_l n_k) dS \quad (k, l = 1, 2, 3),$$

where the index after the comma designates the partial derivative along the corresponding coordinate. Thus,

$$N_{kl}^0 = \frac{1}{2} (U_{kl}^0 + U_{lk}^0) \quad (k, l = 1, 2, 3) \tag{9}$$

(the magnitudes of U_{kl}^0 are determined in (5)). Analogously, for $k \neq i \neq l$ we have

$$\int_v \varepsilon_{kl} x_i dv = \frac{1}{2} \int_S (u_k n_l + u_l n_k) x_i dS,$$

i.e.

$$N^i_{kl} = \frac{1}{2}(U^i_{kl} + U^i_{lk}) \quad (k, l = 1, 2, 3, k \neq i \neq l); \tag{10}$$

for i = k $\neq l$ (there is no summation over k)

$$\int_v \varepsilon_{kl} x_k dv = \frac{1}{2}\int_v (u_{k,l} + u_{l,k})x_k dv = \frac{1}{2}\int_v [(u_k x_k)_{,l} + (u_l x_k)_{,k} -$$

$$- u_l]dv = \frac{1}{2}\int_S (u_k n_l + u_l n_k)x_k dS - \frac{1}{2}\int_v u_l dv;$$

and for i = k = l (there is no summation over l)

$$2N^k_{kl} - N^l_{ll} = U^k_{kl} + U^k_{lk} - U^l_{ll} \quad (k, l = 1, 2, 3, k \neq l).$$

Excluding $\int_v u_l dv$ from these equalities and using symbols (3) and (5), we obtain (there is no summation over k and l)

$$\int_v \varepsilon_{ll} x_l dv = \int_v u_{l,l} x_l dv = \int_S u_l n_l x_l dS - \int_v u_l dv. \tag{11}$$

Thus, formulas (9)-(11) yield 21 relations (6 each from (9), (11) and 9 from (10)), expressing the deformation moments and their combinations by the components of the vector of the shifts of the points on the surface S. Taking equalities (4) into account, these relations form a closed system for determining a_{klmn} from the known tension moments.

Let us write this system in an explicit form. We will introduce the following symbols for the unknown quantities:

$$\alpha_1 = a_{1111}, \alpha_2 = a_{1122}, \alpha_3 = a_{1133}, \alpha_4 = 2a_{1112},$$
$$\alpha_5 = 2a_{1113}, \alpha_6 = 2a_{1123}, \alpha_7 = a_{2222}, \alpha_8 = a_{2233}, \alpha_9 = 2a_{2212}, \alpha_{10} =$$
$$= 2a_{2213}, \alpha_{11} = 2a_{2223}, \alpha_{12} = a_{3333}, \alpha_{13} = 2a_{3312}, \alpha_{14} = 2a_{3313}, \alpha_{15} =$$
$$= 2a_{3323}, \alpha_{16} = 4a_{1212}, \alpha_{17} = 4a_{1213}, \alpha_{18} = 4a_{1223}, \alpha_{19} = 4a_{1313},$$
$$\alpha_{20} = 4a_{1323}, \alpha_{21} = 4a_{2323}.$$

Then, from (4) and (9)-(11) we find

$$A_{pq}\alpha_q = b_p \, (p = 1, 2, \ldots, 21), \tag{12}$$
$$b_1 = U^0_{11}, b_2 = U^0_{22}, b_3 = U^0_{33}, b_4 = U^0_{12} + U^0_{21}, b_5 = U^0_{13} + U^0_{31},$$
$$b_6 = U^0_{23} + U^0_{32}, b_7 = U^2_{11}, b_8 = U^3_{11}, b_9 = U^1_{22}, b_{10} = U^3_{22}, b_{11} = U^1_{33}, b_{12} = U^2_{33},$$
$$b_{13} = U^3_{12} + U^3_{21}, b_{14} = U^2_{13} + U^2_{31}, b_{15} = U^1_{23} + U^1_{32}, b_{16} = U^2_{12} + U^2_{21} - U^1_{11},$$

R.p. 14 $$b_{17} = U^3_{13} + U^3_{31} - U^1_{11}, b_{18} = U^1_{12} + U^1_{21} - U^2_{22}, b_{19} = U^3_{23} + U^3_{32} - U^2_{22},$$

$$b_{20} = U^1_{13} + U^1_{31} - U^3_{33}, b_{21} = U^2_{23} + U^2_{32} - U^3_{33},$$

where the summation over q is carried out from 1 to 21, and the zero elements of the matrix $\|A_{pq}\|$ have the form

$$A_{1.1} = A_{2.2} = A_{3.3} = A_{4.4} = A_{5.5} = A_{6.6} = M^0_{11},$$
$$A_{1.2} = A_{2.7} = A_{3.8} = A_{4.9} = A_{5.10} = A_{6.11} = M^0_{22}, A_{1.3} = A_{2.8} = A_{3.12} = A_{4.13} =$$
$$= A_{5.14} = A_{6.15} = M^0_{33}, A_{1.4} = A_{2.9} = A_{3.13} = A_{4.16} = A_{5.17} = A_{6.18} = M^0_{12},$$
$$A_{1.5} = A_{2.10} = A_{3.14} = A_{4.17} = A_{5.19} = A_{6.20} = M^0_{13},$$

$$A_{1.6} = A_{2.11} = A_{3.15} = A_{4.18} = A_{5.20} = A_{6.21} = M^0_{23},$$
$$A_{7.1} = A_{12.3} = A_{14.5} = -A_{18.2} = -A_{19.2} = A_{21.6} = M^2_{11},$$
$$A_{7.2} = A_{12.8} = A_{14.10} = A_{16.9} = -A_{18.7} = -A_{19.7} = A_{21.11} = M^2_{22},$$

$$
\begin{gathered}
A_{7.3} = A_{12.12} = A_{14.14} = A_{16.13} = -A_{18.8} = -A_{19.8} = M^2_{33}, \\
A_{7.4} = A_{12.13} = A_{14.17} = A_{16.16} = -A_{19.9} = A_{21.18} = M^2_{12}, \\
A_{7.5} = A_{12.14} = A_{14.19} = A_{16.17} = -A_{18.10} = -A_{19.10} = A_{21.20} = M^2_{13}, \\
A_{7.6} = A_{12.15} = A_{14.20} = A_{16.18} = -A_{18.11} = A_{21.21} = M^2_{23}, \\
A_{8.1} = A_{10.2} = A_{13.4} = A_{19.6} = -A_{20.3} = -A_{21.3} = M^3_{11}, \\
A_{8.2} = A_{10.7} = A_{13.9} = A_{17.10} = -A_{20.8} = -A_{21.8} = M^3_{22}, \\
A_{8.3} = A_{10.8} = A_{13.13} = A_{17.14} = A_{19.15} = -A_{20.12} = -A_{21.12} = M^3_{33}, \\
A_{8.4} = A_{10.9} = A_{13.16} = A_{17.17} = A_{19.18} = -A_{20.13} = -A_{21.13} = M^3_{12}, \\
A_{8.5} = A_{10.10} = A_{13.17} = A_{17.19} = A_{19.20} = -A_{21.14} = M^3_{13}, \\
A_{8.6} = A_{10.11} = A_{13.18} = A_{17.20} = A_{19.21} = -A_{20.15} = M^3_{23}, \\
A_{9.2} = A_{11.3} = A_{15.6} = -A_{16.1} = -A_{17.1} = A_{18.4} = A_{20.5} = M^1_{11}, \\
A_{9.7} = A_{11.8} = A_{15.11} = -A_{16.2} = -A_{17.2} = A_{20.10} = M^1_{22}, \\
A_{9.8} = A_{11.12} = A_{15.15} = -A_{16.3} = -A_{17.3} = A_{18.13} = M^1_{33}, \\
A_{9.9} = A_{11.13} = A_{15.18} = -A_{17.4} = A_{18.16} = A_{20.17} = M^1_{12}, \\
A_{9.10} = A_{11.14} = A_{15.20} = -A_{16.5} = A_{18.17} = A_{20.19} = M^1_{13}, \\
A_{9.11} = A_{11.15} = A_{15.21} = -A_{16.6} = -A_{17.6} = A_{18.18} = A_{20.20} = M^1_{23}, \\
A_{16.4} = M^2_{11} - M^1_{12},\ A_{17.5} = M^3_{11} - M^1_{13},\ A_{18.9} = M^1_{22} - M^2_{12}, \\
A_{19.11} = M^3_{22} - M^2_{23},\ A_{20.14} = M^1_{33} - M^3_{13},\ A_{21.15} = M^2_{33} - M^3_{23}.
\end{gathered}
$$

The system of linear equations (12) has a unique solution, if

$$\det \| A_{pq} \| \neq 0. \tag{13}$$

Since the elements of the matrix $\| A_{pq} \|$ are expressed through the moments M_{kl}^i (k, l = 1, 2, 3, i = 0, 1, 2, 3), defined in (7) and (8), inequality (13) is the only condition that the external loads p_k must satisfy so that all of the elastic coefficients of a homogeneous body can be found unambiguously from the known components u_k and p_k on the surface S.

We note that in the derivation of (12), integrals over the volume v were excluded from the shifts. It is not difficult to see that if they are included in the unknowns, then from (4) we obtain a closed system of 24 equations for 24 quantities: α_p (p = 1, 2, ..., 21) and $\int_v u_k dv$ (k = 1, 2, 3). The latter integrals determine the average values of the shifts in the volume v, which are easy to find if the solution to system (12) is known.

The considerations cited above on determination of the strength characteristics from known values of u_k and p_k at the boundary of S can be expanded to more complex bodies: for example, to anisotropic linear tensile-elastic bodies. In this case it is necessary to know the components u_k and p_k on S as functions of time.

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